On the universality of the fluctuation–dissipation ratio in non-equilibrium critical dynamics

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Abstract
The two-time non-equilibrium correlation and response functions in 1D kinetic classical spin systems with non-conserved dynamics and quenched to their zero-temperature critical point are studied. The exact solution of the kinetic Ising model with Glauber dynamics for a wide class of initial states allows for an explicit test of the universality of the non-equilibrium limit fluctuation–dissipation ratio X_∞. It is shown that the value of X_∞ depends on whether the initial state has finitely many domain walls or not and thus two distinct dynamic universality classes can be identified in this model. Generic 1D kinetic spin systems with non-conserved dynamics fall into the same universality classes as the kinetic Glauber–Ising model provided the dynamics is invariant under the C-symmetry of simultaneous spin and magnetic-field reversal. While C-symmetry is satisfied for magnetic systems, it need not be for lattice gases which may therefore display hitherto unexplored types of non-universal kinetics.

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1. Introduction

Understanding the long-time behaviour of strongly interacting systems with many degrees of freedom and evolving far from equilibrium is an active topic of much current interest, see [1–4] for recent reviews. Besides the more far-reaching aspects of disordered systems undergoing glassy behaviour, many of the fundamental questions of non-equilibrium statistical mechanics can already be studied in the conceptually simpler kinetic ferromagnetic systems. In several instances, general ideas can be subjected to exacting tests because several non-trivial and exactly solvable models are available.

In this paper, we consider the kinetics of a purely classical spin system with a non-conserved order parameter and an equilibrium critical temperature $T_c \geqslant 0$ and quenched to a

final temperature $T \leqslant T_c$ from some initial state. Then fluctuations of the initial state will lead on the microscopic level to the growth of correlated domains and the slow movement of the domain boundaries will drive the irreversible time evolution of the macroscopic observables. It has been realized in recent years that the associated ageing effects are more fully revealed through the study of *two-time* correlation functions $C(t, t_w)$ and response functions $R(t, t_w)$ (see section 2 for the precise definitions) and where t is referred to as the observation time while t_w is called the waiting time. In addition, it has been understood that correlation and response functions must be studied together, since out of equilibrium the fluctuation–dissipation theorem is no longer valid. It is usual to characterize the breaking of the fluctuation–dissipation theorem in terms of the *fluctuation–dissipation ratio* (FDR) [5]

$$X(t, t_w) := T R(t, t_w) \left(\frac{\partial C(t, t_w)}{\partial t_w} \right)^{-1} \tag{1.1}$$

where at equilibrium, one would recover $X(t, t_w) = 1$ (at $T = 0$, some care is needed to absorb T into the definition of the response function). In the scaling limit, where simultaneously t_w *and* $t - t_w$ become large, one actually finds that the fluctuation–dissipation ratio $X(t, t_w) = \hat{X}(t/t_w)$ satisfies a simple scaling law in terms of the scaling variable $x = t/t_w$, see [4] for a recent review. A quantity of particular interest is the *limit fluctuation–dissipation ratio X_∞* defined by

$$X_\infty = \lim_{t_w \to \infty} (\lim_{t \to \infty} X(t, t_w)) = \lim_{x \to \infty} \hat{X}(x). \tag{1.2}$$

For a quench into the disordered phase, it is known that $X_\infty = 0$ in general. But for a critical quench with $T = T_c$, Godrèche and Luck [6, 7] have proposed that X_∞ should be a *universal* quantity.

The evidence supporting this conjecture (which in this paper we shall call '*universality*' for short) was based on the available results coming from exactly solvable kinetic spin systems quenched from a fully disordered state and from simulations in the 2D and 3D kinetic Ising models with Glauber dynamics, see [2] for a review. Further supporting evidence in favour of the universality conjecture comes from field-theoretic two-loop calculations of the $O(n)$-model, again starting from a fully disordered initial state [8, 9]. The universality of X_∞ has also been confirmed numerically for the 2D Glauber–Ising and voter models [10].

Statements about the universality of a physical quantity are best tested by varying important control parameters of suitable models and then studying their effects. Indeed, the role of spatially long-range initial correlations of the form

$$C_{\text{ini}}(\boldsymbol{r}) \sim |\boldsymbol{r}|^{-\nu} \tag{1.3}$$

where ν is a control parameter, was studied in the kinetic spherical model at $T = T_c$ [11]. It was shown that there exists an unexpectedly rich kinetic phase diagram, depending on ν and the space dimension d. In most of these phases, either $X_\infty = 0$ or else it is independent of ν, but in one phase X_∞ was shown to be a function of ν [11], thus furnishing an important qualification against the unrestricted universality of X_∞.[4] At present, it is not clear yet whether these results might not simply reflect a peculiarity of the spherical model. Therefore, we shall study here the non-equilibrium critical dynamics of 1D ferromagnetic spin systems quenched to their critical temperature $T = 0$.

In order to get analytical results, we consider in section 2 the exactly solvable Glauber–Ising model. We shall show that for initial correlations of the form (1.3) with $\nu \geqslant 0$, we have indeed universality of the entire function $\hat{X}(x)$, and thus in particular of X_∞, in agreement

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[4] In this phase, both the space dimension d and the effective dimension $D = 2 + \nu$ of the initial correlations are below the upper critical dimension d^*.

with the Godrèche–Luck conjecture. However, in section 3 we study even more general initial states which consist of large ordered domains and then show that the scaling function $\hat{X}(x)$ as well as X_∞ is different from those obtained for initial states of the form (1.3). We thereby identify two dynamic universality classes. These results are extended in section 4 to generalized two-state spin systems evolving according to a non-conserved dynamics with detailed balance. We show that those which satisfy a certain global symmetry (which we call C-symmetry) are in one of the two universality classes found for the Glauber–Ising model. We also comment on the fact that in lattice gases, there is no need to satisfy C-symmetry which may lead to further non-universalities. We conclude in section 5.

2. Glauber dynamics with correlated initial conditions

We shall study the Ising chain with equilibrium Hamiltonian $\mathcal{H} = -J \sum_n s_n s_{n+1}$ where $s_n = \pm 1$ denotes the Ising spins and $J > 0$ is the interaction strength. We consider translation-invariant initial distributions such that the initial magnetization $m_0 = \langle s_n \rangle$ should be different from ± 1, i.e. there is a *finite density* of domain walls at initial time. In the literature, usually symmetric initial distributions with $m_0 = 0$ are considered. In this section, we allow for general non-symmetric initial distributions with $-1 < m_0 < 1$. The case $|m_0| = 1$ requires separate treatment and is studied in section 3. In the absence of a magnetic field we assume Glauber dynamics [12] for the stochastic time evolution of the spins. We set the time scale for individual spin-flips to unity.

2.1. Two-time correlation function

It is convenient to write the two-time correlation function

$$C_n(t + \tau, t) = C_n(t; \tau) := \langle s_n(t + \tau) s_0(t) \rangle \tag{2.1}$$

in the quantum Hamiltonian formalism, see [13, 14] for recent reviews. One has

$$C_n(t; \tau) = \langle s | \sigma_n^z \exp(-H_0 \tau) \sigma_0^z \exp(-H_0 t) | P_0 \rangle \tag{2.2}$$

where $\sigma_n^{x,y,z}$ are the Pauli matrices acting on the nth site of the chain and

$$H_0 = \frac{1}{2} \sum_n \left(1 - \sigma_n^x\right) \left(1 - \frac{\gamma}{2} \sigma_n^z \left(\sigma_{n-1}^z + \sigma_{n+1}^z\right)\right) \tag{2.3}$$

is the Markov generator (stochastic Hamiltonian) for Glauber dynamics [15]. Furthermore, $|P_0\rangle$ is the probability vector representing the initial distribution of spins and the constant summation vector $\langle s|$ is the left steady state. The generator is constructed such that it satisfies detailed balance with respect to the equilibrium distribution at temperature T of the 1D zero-field ferromagnetic Ising model with interaction strength J. This is achieved by setting $\gamma = \tanh(2J/T)$ (we use units such that the Boltzmann constant $k_B = 1$). We introduce the shorthand

$$C_n := C_n(0, 0) \qquad C_n(t) := C_n(t; 0) = \langle s_n(t) s_0(t) \rangle \tag{2.4}$$

for the initial correlations and for the equal-time correlation function, respectively. Note that

$$C_n(t) = C_{-n}(t) \qquad C_0(t) = 1 \qquad \forall t \geqslant 0. \tag{2.5}$$

For Glauber dynamics the time evolution of the spin-expectation is linear and at vanishing temperature $T = 0$ satisfies a lattice diffusion equation for a 1D random walk with hopping rate 1/2 [12]. The propagator

$$G_n(y) = e^{-y} I_n(y) \tag{2.6}$$

of the lattice diffusion equation involves the modified Bessel function $I_n(y)$ [16] and describes the probability of moving a distance of n lattice units after time y. Hence

$$\langle s|\sigma_n^z \exp(-H_0\tau) = \sum_{m=-\infty}^{\infty} e^{-\tau} I_{n-m}(\tau)\langle s|\sigma_m^z \tag{2.7}$$

which immediately yields

$$C_n(t;\tau) = \sum_{m=-\infty}^{\infty} e^{-\tau} I_{n-m}(\tau) C_m(t). \tag{2.8}$$

It remains to calculate the equal-time two-point correlation function $C_m(t)$. For special initial distributions this has already been done in the classical paper by Glauber [12]. For our more general treatment we observe that the total correlation function may be split into an interaction part $C_m^{\text{int}}(t)$ and a correlation part $C_m^{\text{corr}}(t)$. The latter vanishes for uncorrelated (infinite-temperature) initial states. Then

$$C_m(t) = e^{-2t} I_m(2t) + 2\sum_{k=1}^{\infty} e^{-2t} I_{|m|+k}(2t) + \sum_{k=1}^{\infty} e^{-2t}[I_{|m|-k}(2t) - I_{|m|+k}(2t)]C_k$$
$$=: C_m^{\text{int}}(t) + C_m^{\text{corr}}(t). \tag{2.9}$$

The two-time autocorrelation function then follows from (2.8) by setting $n = 0$.

The Laplace transform of the interaction part of the two-time autocorrelation function has already been studied in detail by Godrèche and Luck [6] and by Lippiello and Zannetti [17][5]. Here, we prefer to work in the more intuitive time domain, which allows for an easier analysis of more general initial conditions. From (2.9), we also define $C_n^{\text{ini}}(t;\tau)$ and $C_n^{\text{corr}}(t;\tau)$ which will be calculated separately and $C_n(t;\tau) = C_n^{\text{ini}}(t;\tau) + C_n^{\text{corr}}(t;\tau)$. Using the completeness property $\sum_n G_n = 1$ of the lattice propagator we split the interaction part into a contribution which is large at early times and a second contribution which dominates the late-time behaviour. We find

$$C_0^{\text{int}}(t;\tau) = \exp(-(\tau+2t))I_0(\tau+2t) + e^{-\tau} I_0(\tau)[1 - e^{-2t} I_0(2t)]$$
$$+ 4\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} e^{-2t} I_{|m|+k}(2t) e^{-\tau} I_m(\tau). \tag{2.10}$$

For $t, \tau \gg 1$ only the late-time part (containing the double sum) plays a role. Using the asymptotic Gaussian form

$$G_n(y) = \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{n^2}{2y}\right)(1 + \mathrm{O}(y^{-1})) \tag{2.11}$$

of the lattice propagator we can turn the sums into integrals. Setting

$$\alpha = \sqrt{\frac{\tau}{2t}} \tag{2.12}$$

we obtain

$$C_0^{\text{int}}(t;\tau) = \frac{4}{\pi} \int_0^{\infty} du \int_0^{\infty} dv \exp(-u^2 + (u\alpha+v)^2)$$
$$= 1 - \frac{2}{\pi} \arctan\alpha = \frac{2}{\pi} \arctan\frac{1}{\alpha}. \tag{2.13}$$

The correlation function depends only on the scaling variable α.

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[5] Multispin correlators and associated response functions of the 1D Glauber–Ising model have been studied recently in [18, 19]. Two-time correlators of the Ising chain in a transverse field are calculated in [20].

For later comparison with the response function, we now make a small change in notation, in order to be compatible with the notation usually employed [6, 11, 17]. The quantity denoted 'time' t so far, we shall from now on call *waiting time* $s = t_w$. In turn, we call $t_w + \tau$ *observation time* and write $t := t_w + \tau$. We repeat the correlation function in this notation, in agreement with [6, 17]

$$C_0^{\text{int}}(t, t_w) = \frac{2}{\pi} \arctan \sqrt{\frac{2t_w}{t - t_w}}. \tag{2.14}$$

For later use, we also record the derivative of C_0^{int} with respect to t_w. Keeping t fixed, we have

$$-\frac{\partial}{\partial t_w} C_0^{\text{int}}(t, t_w) = -\frac{1}{2\pi} \frac{1 + 2\alpha^2}{\alpha(1 + \alpha^2)} \cdot \frac{1}{t_w} = \frac{\sqrt{2}}{\pi} \frac{x}{\sqrt{x - 1}(x + 1)} \cdot \frac{1}{t_w} \tag{2.15}$$

for either the scaling variable α or $x = t/t_w$.

We still have to analyse the correlation part of the autocorrelation function which has not been studied previously. In order to do so, we use the integral representation

$$G_n(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \cos(qn) \exp(-\epsilon_q y) \tag{2.16}$$

of the lattice propagator with the dispersion relation $\epsilon_q = (1 - \cos q)$. This yields the exact expression

$$C_0^{\text{corr}}(t_w; \tau) = \sum_{m=-\infty}^{\infty} e^{-\tau} I_m(\tau) C_m^{\text{corr}}(t_w)$$

$$= \sum_{m=1}^{\infty} G_m(\tau) \sum_{n=1}^{\infty} C_n \frac{2}{\pi} \int_{-\pi}^{\pi} dq \sin(qm) \sin(qn) \exp(-\epsilon_q t_w). \tag{2.17}$$

For $t_w, \tau \gg 1$ the sum over m may be turned into an integral and the asymptotic expression (2.11) can be used. After some algebra we find

$$C_0^{\text{corr}}(t_w; \tau) = \frac{4\alpha}{\pi^{3/2} t_w^{1/2}} \sum_{n=1}^{\infty} C_n \int_0^{\pi} dq \, q \sin\left(\frac{qn}{\sqrt{t_w}}\right) {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \alpha^2 q^2\right) \exp(-q^2(1 + \alpha)^2) \tag{2.18}$$

where ${}_1F_1(a; b; x)$ is a confluent hypergeometric series [16]. In order to analyse the leading behaviour for waiting times $t_w \gg 1$ we have to distinguish three cases, which depend on the form of the initial correlation. We shall in this section consider the following form:

$$C_n(0, 0) = C_n \sim \frac{B}{n^{\nu}} \qquad \text{for} \quad n \to \infty \tag{2.19}$$

where $\nu \geqslant 0$ and B are—*a priori* non-universal—control parameters. We also define the unnormalized first moment

$$A := \sum_{n=1}^{\infty} n C_n \tag{2.20}$$

of the initial correlation function. We can now identify three distinct situations.

Case 1: $A < \infty$. Consider first the situation when the series A converges to a *finite* value. Physical examples might be antiferromagnetic alternating-sign correlations or rapidly decaying ferromagnetic correlations with $\nu > 2$. For large waiting times t_w the leading contribution to

the integral in (2.18) comes from small values of the integration variable q (which describe the long wavelength fluctuations of the spin system). We may then expand the sine in its Taylor series and the integration can be performed explicitly from the series representation of the hypergeometric function. Summing up the infinite series of Gaussian integrals yields the surprisingly simple exact asymptotic expression, to leading order in t_w

$$C_0^{\text{corr}}(t_w, \tau) = \frac{A}{\pi} \frac{\alpha}{1 + \alpha^2} \cdot \frac{1}{t_w}. \tag{2.21}$$

Because of the extra factor $1/t_w$, this is asymptotically smaller than the interaction part (2.13). Hence the non-universal amplitude A does not contribute to the leading late-time asymptotics of the two-time autocorrelation function.

Case 2: Slowly decaying ferromagnetic correlations. We now consider the initial correlator (2.19) with $0 < \nu < 2$ (such that A does not converge). Replacing the summation over n by an integration

$$\frac{1}{\sqrt{t_w}} \sum_{n=1}^{\infty} C_n \sin\left(\frac{qn}{\sqrt{t_w}}\right) \rightarrow \int_0^{\infty} dy\, C(y\sqrt{t_w}) \sin qy \tag{2.22}$$

$$= Bt_w^{-\nu/2}\Gamma(1 - \nu) \cos\left(\frac{\pi\nu}{2}\right)|q|^{\nu-1}\text{sign}(q). \tag{2.23}$$

This yields

$$C_0^{\text{corr}}(t_w; \tau) = \frac{B2^{1-\nu}}{\pi} \Gamma\left(1 - \frac{\nu}{2}\right) \frac{\alpha}{(1 + \alpha^2)^{(1+\nu)/2}} {}_2F_1\left(\frac{1}{2}, \frac{1+\nu}{2}; \frac{3}{2}; \frac{\alpha}{1+\alpha^2}\right) \cdot t_w^{-\nu/2}. \tag{2.24}$$

Also in this case the correlation part of the two-time autocorrelation function is asymptotically small compared to the interaction part. We conclude that the non-universal quantities B and ν, which characterize the initial distribution before the quench, do not enter into the leading late-time behaviour.

Case 3: Partial ferromagnetic long range order. The case $\nu = 0$ corresponds to an asymptotically constant spin–spin correlation function in the initial state, mimicking (partial) ferromagnetic long range order. Such initial states may, for example, be obtained by quenching from a uniformly magnetized initial state to zero temperature and zero field and have already been studied in [11] where $B = m_0^2$ was related to the initial magnetization. Performing the same steps as in case 2 we find

$$C_0^{\text{corr}}(t_w; \tau) = \frac{2B}{\pi} \arctan \alpha. \tag{2.25}$$

This is of the same order of magnitude as the interaction contribution to the correlation function. For the total correlation function we obtain

$$C_0(t_w; \tau) = 1 - (1 - B)\frac{2}{\pi} \arctan \alpha \tag{2.26}$$

and therefore, for t fixed we have from (2.15)

$$-\frac{\partial}{\partial t_w} C_0(t, t_w) = \frac{1 - B}{\pi t_w} \frac{x}{1 + x} \sqrt{\frac{2}{x - 1}} \tag{2.27}$$

in terms of the scaling variable $x = t/t_w$. This is of the same form as in the uncorrelated case, but with a non-universal amplitude $1 - B$ determined by the initial long range order. The form (2.27) proves universality with regard to details of the initial distribution of the two-time correlation function, except for a non-universal amplitude.

2.2. Two-time response function

Now we consider the time evolution of the local magnetization

$$S_n(t) = \langle \sigma_n^z(t) \rangle = \langle s | \sigma_n^z \exp(-Ht) | P_0 \rangle \qquad (2.28)$$

for an initial distribution with initial magnetization $m_0 \neq \pm 1$. In zero field $S_n(t) = m_0$ for $T = 0$ Glauber dynamics. To study the linear response of the system to a small localized perturbation by a magnetic field we let an external field act at site 0 of the lattice. In the quantum Hamiltonian formulation this perturbation of the zero-field dynamics is represented by the perturbed Markov generator

$$H = H_0 + V(h). \qquad (2.29)$$

The perturbation $V(h)$ is determined by the requirement that the full generator H satisfies detailed balance with respect to the equilibrium distribution

$$P^* \sim \exp\left[\frac{1}{T} \sum_n (Js_n s_{n+1} + hs_0) \right] \qquad (2.30)$$

of the ferromagnetic Ising system with interaction strength J and local magnetic field h at site 0. This requirement, on which the usual equilibrium fluctuation–dissipation theorem is based, does not uniquely fix V, as different dynamical rules may lead to the same equilibrium distribution (2.30). In [6] a heat-bath prescription was used to implicitly define V. Here we follow more closely the philosophy of Glauber [12] and define a minimally perturbed dynamics by

$$V = \frac{1}{2}(1 - \sigma_0^x)\left[1 - \frac{\gamma}{2}\sigma_0^z(\sigma_{-1}^z + \sigma_1^z) \right] \left[\exp(-(h/T)\sigma_0^z) - 1 \right]. \qquad (2.31)$$

At zero temperature one has $\gamma = 1$. We shall use the dimensionless field strength h/T throughout this work.

Following standard procedures we let the field act at time t_w and calculate the linear response function (in units of T)

$$R_n(t, t_w) = R_n(t_w; \tau) = \frac{\delta}{\delta h(t_w)} S_n(t) \qquad (2.32)$$

at observation time t. As before $\tau = t - t_w \geqslant 0$ is the time elapsed after the perturbation. By expanding the full time evolution operator $\exp(-Ht)$ in powers of h we find from (2.28), (2.32)

$$R_n(t_w; \tau) = -\langle s | \sigma_n^z \exp(-H\tau) V' \exp(-Ht_w) | P_0 \rangle. \qquad (2.33)$$

Here V' is the derivative of V with respect to h/T taken at $h = 0$. Using (2.7) we see after a little algebra that the autoresponse function ($n = 0$) factorizes

$$R_0(t_w; \tau) = e^{-\tau} I_0(\tau) [1 - C_1(t_w)1] \qquad (2.34)$$

into the autopropagator $G_0(\tau)$ and a contribution involving the two-point correlation function at time t_w.

To calculate the interaction part of the response function we deduce by analogy with (2.10)

$$1 - C_1^{\text{int}}(t) = e^{-2t}(I_0(2t) + I_1(2t)). \qquad (2.35)$$

For large times $t_w \gg 1$ we then find

$$R_0^{\text{int}}(t_w; \tau) = \frac{1}{\sqrt{2\pi\tau}} \frac{2}{\sqrt{\pi t_w}} = \frac{1}{\pi t_w} \frac{1}{\alpha}. \qquad (2.36)$$

With $\alpha^2 = (x - 1)/2$ one obtains the autoresponse function in terms of the scaling variable $x = t/t_w$. Interestingly, the same asymptotic result was found in [6] for heat-bath dynamics.

The calculation of the correlation part of the autoresponse function proceeds along the same lines as the calculation of the autocorrelation function. We find

Case 1: $A < \infty$. Here

$$C_1^{\text{corr}}(t_w) = \frac{A}{4\pi^{1/2}t_w^{3/2}} \tag{2.37}$$

for large t_w. Comparison with (2.35) shows that we have a subleading behaviour of the correlation contribution.

Case 2: Weakly decaying ferromagnetic correlations. Computing the correlation as above leads to

$$C_1^{\text{corr}}(t_w) = \frac{B}{2^\nu \sqrt{\pi}} \Gamma \left(1 - \frac{\nu}{2}\right) t_w^{-(\nu+1)/2} \tag{2.38}$$

which corresponds again to subleading behaviour. Hence initial correlations decaying to zero do not change the asymptotic behaviour of the autoresponse function.

Case 3: Ferromagnetic long range order. For $\nu = 0$ (correlations decaying to a constant value B) we obtain

$$C_1^{\text{corr}}(t_w) = \frac{B}{\sqrt{\pi t_w}} \tag{2.39}$$

which is of the same order as the interaction part. Therefore,

$$R_0(t_w; \tau) = \frac{(1 - B)}{\pi \sqrt{2\tau t_w}} = \frac{1 - B}{2\pi t_w} \frac{1}{\alpha}. \tag{2.40}$$

We see from equations (2.27), (2.40) that the same non-universal amplitude enters the two-time correlation function and the response function, respectively.

We can now state the main result of this section: in each of the cases 1–3 the fluctuation–dissipation ratio $X = R/\dot{C}$ does not depend on the initial state and is given by

$$X(t, t_w) = R(t, t_w) \left(\frac{\partial C(t, t_w)}{\partial t_w}\right)^{-1} = \hat{X}(x) = \frac{x + 1}{2x}. \tag{2.41}$$

The same result was obtained in [6] for different microscopic dynamics and uncorrelated initial states. We note in passing that one may easily check that also for complete initial antiferromagnetic order $C_n = (-1)^n$ the FDR has the same asymptotic form. In the limit $x \to \infty$ we find

$$X_\infty = \lim_{x \to \infty} \hat{X}(x) = \tfrac{1}{2} \tag{2.42}$$

in full agreement with the conjecture [6, 7] that X_∞ is a universal constant.

3. Low-temperature initial states

In the previous section, we assumed a translation-invariant state with a *finite density* of domain walls in the Ising system at the initial time. However, at very low temperatures it is more relevant to study the time evolution of an almost ordered system with only *finitely many* domain walls at the initial time. For definiteness we consider two domain walls located at sites $-L$ and L respectively of the lattice. This corresponds to the initial configuration

$$P_0 = \cdots \downarrow\downarrow\downarrow\uparrow\uparrow \cdots \uparrow\uparrow\downarrow\downarrow \cdots \tag{3.1}$$

where the inversions of the spins occur at the positions $-L$ and $+L$, respectively. Although these initial conditions break translation invariance, we have chosen the coordinate system such that reflection symmetry with respect to the origin is maintained. This is not crucial, but simplifies some of the exact expressions to be derived below.

In order to calculate the two-time correlation function we use the enantiodromy relation

$$H_0^T = B H_A B^{-1} \tag{3.2}$$

between Glauber dynamics and diffusion-limited pair annihilation (DLPA) [13, 21][6]. The process DLPA describes independent random walkers hopping with rate $1/2$ to a nearest-neighbour site and annihilating instantaneously upon encounter. Here H_0^T is the transpose of the Hamiltonian for zero-temperature Glauber dynamics, H_A is the Markov generator for DLPA and B is the factorized similarity transformation $B = b^{\otimes N}$ with the local transformation matrix

$$b = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}. \tag{3.3}$$

With the enantiodromy relation (3.2) and the initial state (3.1) one obtains in the thermodynamic limit $N \to \infty$

$$C_{m,n}(t) = \langle s | \sigma_m^z \sigma_n^z \exp(-H_0 t) | P_0 \rangle = \langle s | \prod_{k=-L}^{L} \sigma_k^z \exp(-H_A t) | m, n \rangle. \tag{3.4}$$

By identifying spin up with a vacancy and spin down with a particle in the process of diffusion-limited annihilation the vector $|m, n\rangle$ is the state with two particles located at sites m, n and empty sites everywhere else [13]. Hence the calculation of the two-point correlation function is reduced to a two-particle problem of annihilating particles.

The two-time autocorrelation function at site $n = 0$ is given by (2.8) in terms of the equal-time correlation function. By reflection symmetry one has $C_{-m,0}(t) = C_{0,m}(t)$ and therefore we may write

$$C_0(t_w; \tau) = 1 - 2 \sum_{m=1}^{\infty} e^{-\tau} I_m(\tau)[1 - C_{0,m}(t_w)]. \tag{3.5}$$

With (3.4) one has

$$1 - C_{k,l}(t) = \sum_{x=-\infty}^{\infty} \sum_{y=x+1}^{\infty} \langle s | \left(1 - \prod_{k=-L}^{L} \sigma_k^z \right) | x, y \rangle P(x, y; t | k, l; 0) \tag{3.6}$$

where the two-particle propagator

$$P(x, y; t | k, l; 0) = e^{-2t} [I_{k-x}(t) I_{l-y}(t) - I_{k-y}(t) I_{l-x}(t)] \tag{3.7}$$

for DLPA is the probability that two independent random walkers who started at k, l have reached sites x, y after time t without having met on the same site. This yields

$$1 - C_{0,m}(t) = 2 \sum_{y=-L}^{L} e^{-2t} I_y(t) \sum_{x=L+1-m}^{L+m} I_x(t). \tag{3.8}$$

We are interested in the derivative $\partial C(t, t_w)/\partial t_w$ for large t. For an initial size $2L + 1$ of the domain of flipped spins we consider the regime where $t_w \gg L^2$ since at earlier times fluctuations in the initial positions of the domain walls have not reached the origin and hence

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[6] This relation is not to be confused with the duality relation [22] between Glauber dynamics and diffusion-limited pair annihilation.

would lead to trivial behaviour. To calculate the derivative of the correlator we express the exact expression (3.5) in terms of the variables t_w, t as in (2.14). Some algebra similar to the previous sections yields for fixed t

$$-\frac{\partial}{\partial t_w} C_0(t, t_w) = \frac{2L+1}{\sqrt{2\pi^3 t_w^3}} \left(\frac{1}{\sqrt{2}\alpha} + \arctan(\sqrt{2}\alpha) \right). \tag{3.9}$$

With (2.31), (2.33) the autoresponse function is given by

$$R(t_w; \tau) = G_0(\tau) \frac{1}{2} (2 - C_{0,1} - C_{-1,0}(t_w))$$

$$= G_0(\tau) [G_L(t_w) + G_{L+1}(t_w)] \sum_{y=-L}^{L} G_y(t_w) \tag{3.10}$$

where we used (3.6). In the regime $t_w \gg L^2$ this reduces to

$$R(t_w; \tau) = \frac{2L+1}{\sqrt{2\pi^3}} \frac{1}{\sqrt{2}\alpha} \cdot t_w^{-3/2}. \tag{3.11}$$

As expected both the autocorrelation function and the autoresponse function contain the non-universal amplitude $2L + 1$ which is the initial size of the flipped domain. However, the FDR is a universal function of the scaling variable $x = 1 + 2\alpha^2$. We find

$$X(t, t_w) = \hat{X}(x) = \frac{1}{1 + \sqrt{x-1} \arctan\left(\sqrt{x-1}\right)}. \tag{3.12}$$

This scaling function is different from (2.41), in particular

$$X_\infty = 0 \tag{3.13}$$

in contradiction to the universality hypothesis for critical dynamics as formulated in [6].

It might be helpful to restate our results in terms of scaling functions. In the ageing regime, where both t_w and $t - t_w$ become large, one expects *at* criticality, see e.g. [2]

$$C(t, s) = s^{-a} f_C(t/s) \qquad R(t, s) = s^{-1-a} f_R(t/s) \tag{3.14}$$

where a is a non-equilibrium exponent and such that for large arguments $x \to \infty$, $f_{C,R}(x) \sim x^{-\lambda_{C,R}/z}$ which defines the autocorrelation and autoresponse exponents λ_C and λ_R, respectively. This implies that $\hat{X}(x) \sim x^{(\lambda_C - \lambda_R)/z}$ for $x \to \infty$. The dynamical exponent $z = 2$ throughout in the model at hand but the other exponents depend on the initial conditions as follows. If we take an initial state with decaying correlation of the power-law form (2.22), we read off from the results of section 2

$$a = 0 \qquad \lambda_C = 1 = \lambda_R. \tag{3.15}$$

However, for an initial state of the form (3.1), we find

$$a = \tfrac{1}{2} \qquad \lambda_C = 0 \qquad \lambda_R = 1. \tag{3.16}$$

We therefore see explicitly that the different forms of the scaling function $\hat{X}(x)$ signal two distinct dynamical universality classes.

4. C-violation and non-universality

In the previous sections, we considered special spin-flip dynamics which in zero field reduces to Glauber dynamics. The most general flip rates for a local magnetic field which (i) satisfy

detailed balance with respect to the equilibrium distribution (2.30), (ii) correspond to reflection-symmetric finite-range interactions and (iii) are non-conserved and in particular lead to Glauber rates for $h = 0$ are as follows:

$$
\begin{aligned}
\uparrow\uparrow\uparrow &\rightarrow \uparrow\downarrow\uparrow \text{ with rate } (1 - \gamma) f_1 \exp(-h/T) \\
\uparrow\downarrow\uparrow &\rightarrow \uparrow\uparrow\uparrow \qquad\qquad (1 + \gamma) f_1 \exp(h/T) \\
\uparrow\uparrow\downarrow &\rightarrow \uparrow\downarrow\downarrow \qquad\qquad f_2 \exp(-h/T) \\
\uparrow\downarrow\downarrow &\rightarrow \uparrow\uparrow\downarrow \qquad\qquad f_2 \exp(h/T) \\
\downarrow\uparrow\uparrow &\rightarrow \downarrow\downarrow\uparrow \qquad\qquad f_2 \exp(-h/T) \\
\downarrow\downarrow\uparrow &\rightarrow \downarrow\uparrow\uparrow \qquad\qquad f_2 \exp(h/T) \\
\downarrow\uparrow\downarrow &\rightarrow \downarrow\downarrow\downarrow \qquad\qquad (1 + \gamma) f_3 \exp(-h/T) \\
\downarrow\downarrow\downarrow &\rightarrow \downarrow\uparrow\downarrow \qquad\qquad (1 - \gamma) f_3 \exp(h/T).
\end{aligned}
$$

Here $f_i = f_i(h/T)$ which may also depend on the neighbouring spin variables and must be such that $f_i(0) = 1$. With these functions the flip rates at site 0 may be written as

$$
\begin{aligned}
w(h) &= \left[1 - \frac{\gamma}{2} s_0(s_{-1} + s_1)\right] \exp(-h/T s_0) \\
&\quad \times [f_1 + 2f_2 + f_3 + (f_1 - f_3)(s_{-1} + s_1) + (f_1 - 2f_2 + f_3)s_{-1}s_1 + \text{SRT}] \\
&= \left[1 - \frac{\gamma}{2} s_0(s_{-1} + s_1)\right] \exp(-h/T s_0)[g_0 + g_1(s_{-1} + s_1) + g_2 s_{-1}s_1 + \text{SRT}].
\end{aligned} \tag{4.1}
$$

The short range terms SRT (which vanish for pure nearest-neighbour interactions) involve lattice sites at distances $|k| > 1$ from the origin. Glauber dynamics [12] corresponds to $g_0 = 1, g_1 = g_2 = 0$. With these rates the Markov generator H takes the form $H = \sum_n h_n$ with the local spin flip generators h_n defined by (2.3) for $n \neq 0$ and

$$
h_0 = \tfrac{1}{2}\left(1 - \sigma_0^x\right)\hat{w}(h). \tag{4.2}
$$

Here $\hat{w}(h)$ is the diagonal matrix obtained from (4.1) by replacing the classical spin variables s_i by the Pauli matrices σ_i^z [13].

To first order in h the perturbation V of the stochastic time evolution that corresponds to this general choice of rates is obtained from the expansion of $\hat{w}(h)$ to first order in h. This leads to the following general form of the response function:

$$
R(t_w; \tau) = G_0(\tau) \left\langle \left[s_0 - \frac{\gamma}{2}(s_{-1} + s_1)\right] [s_0 - g_0' - g_1'(s_{-1} + s_1) - g_2' s_{-1}s_1 - \cdots] \right\rangle \tag{4.3}
$$

where the correlation function is evaluated at time t_w. The dots denote contributions from the SRT next-nearest-neighbour interactions and the constants g_i' are defined by $g_i' = T\partial g_i/\partial h|_{h=0}$. Glauber dynamics as used in the previous sections corresponds to the choice $g_i' = 0$ [12]. The heat-bath dynamics of [6] corresponds to the choice $g_0' = g_2' = 0, g_1' = -\gamma/2$.

4.1. C-invariance

In the context of spin systems the stochastic dynamics defined by the rate functions f_i must remain invariant under simultaneous reversal of all spins $s_k \rightarrow -s_k$ and change of sign in the external magnetic field. This global symmetry is an automorphism on the state space analogous to charge conjugation in the quantum field theory of elementary particles. Because of this analogy, we shall refer to it as *C-symmetry*. C-symmetry of the rates requires the

following properties:

$$f_1(h) = f_3(-h) \tag{4.4}$$

$$f_2(h) = f_2(-h) \tag{4.5}$$

of the rate functions. Therefore $g_0'(0) = g_2'(0) = 0$ for C-invariant systems.

For translation-invariant initial distributions the response function (4.3) at $T = 0$ (corresponding to $\gamma = 1$) then takes the form

$$R^{(C)}(t_w; \tau) = G_0(\tau) \left[1 - C_1(t_w) + g_1' \frac{\mathrm{d}}{\mathrm{d}t_w} C_1(t_w) - \left\langle \left(s_0 - \frac{1}{2}(s_{-1} + s_1) \right) (\cdots) \right\rangle \right] \tag{4.6}$$

where equations (2.2), (2.3) were used. The dots denote further SRT terms coming from unspecified non-nearest-neighbour short range interactions. For long times the first contribution (proportional to $1 - C_1(t)$) contains no non-universal parameter of the dynamics. This is indeed the leading contribution since the second contribution, being a time derivative with respect to t_w, is subleading. Finally, the third term is a second-order (lattice) partial space derivative which by dynamical scaling is of the same subleading order in time as a time derivative.

4.2. Consequences for non-universality

The C-symmetry is a physical requirement that any spin dynamics must satisfy. However, the Ising model may also be regarded as a classical lattice gas model, see e.g. [23, 24]. In this interpretation the equilibrium distribution (2.30) describes a system of hard-core particles with attractive nearest-neighbour interactions which may occupy the sites of a lattice, as indicated by the occupation numbers $n_i = (1 - s_i)/2 \in \{0, 1\}$. The external magnetic field then corresponds to a chemical potential and a spin-flip corresponds to particle/vacancy interchange, i.e., an exchange of particles with an external reservoir. Clearly, in this interpretation of the same model there is no need to enforce the constraints (4.4), (4.5). Hence it is interesting to investigate universality in the absence of this symmetry.

The response function contains the same universal part $R^{(C)}$ from equation (4.6) as in the symmetric case plus two further terms

$$R(t_w; \tau) = R^{(C)}(t_w; \tau) - G_0(\tau)[(1 - \gamma)m_0 g_0' - (\gamma m_0 - \langle s_1(t_w)s_0(t_w)s_1(t_w) \rangle)g_2']. \tag{4.7}$$

The first term in the second line vanishes at $T = 0$ since then $\gamma = 1$. The second term is non-zero only for symmetric initial states with $m_0 = 0$. As a function of time it is of the same order as $R^{(C)}(t_w; \tau)$.

5. Conclusions

Our study of universality in the critical, non-conserved, dynamics of purely classical quenched lattice models has uncovered some of the basic mechanisms behind the universality of the scaling of correlation and response functions. Explicit calculations for 1D kinetic spin systems with non-conserved dynamics and which generalize the Glauber–Ising model have shown that the asymptotic form of the two-time response function does *not* depend on the microscopic form of the stochastic dynamics, provided only that C-symmetry holds[7]. Furthermore, in the scaling limit $t_w \gg 1$ and $t - t_w \gg 1$ both the response function and the time derivative of the

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[7] For the 1D Glauber–Ising model, we have explicitly shown that our realization of Glauber dynamics reproduces the same results as the heat-bath dynamics studied in [6, 17].

associated two-time correlation are largely independent of the form of the correlations in the initial state, in agreement with the expected universality of the fluctuation–dissipation ratio $X = \hat{X}(x)$ and its limit values $X_\infty = \lim_{x \to \infty} \hat{X}(x)$. This universality holds true with respect to the long-range decay of the initial correlations (see equation (1.3) with $\nu \geqslant 0$) and also with respect to the 'details' of the dynamics. On the other hand, we have also shown the existence of two *distinct* kinetic universality classes which arise for initial magnetization $|m_0| = 1$ and $|m_0| < 1$, respectively.

In the absence of C-invariance these results only hold true for symmetric initial states with $m_0 = 0$. For $m_0 \neq 0$, a non-universal part will contribute to the long-time behaviour of two-time correlation and response functions. Since lattice gases are in general not C-invariant, a study of these systems will make such terms apparent. In this context, recent attempts [25, 26] to construct algorithms which allow us to measure the two-time response directly, may be of value.

Finally, the available exactly solvable models (the 1D Glauber–Ising model and the kinetic mean spherical model) have revealed two distinct possible routes towards modifications of critical dynamics beyond a fully disordered initial state: (1) the presence of large ordered domains and (2) the interplay of strong initial fluctuations with the thermal fluctuations of the bulk. However, in these models, the two-point functions decouple and can be studied independently of any longer-range correlations. A test of our scenario of possible non-universalities in more general systems, such as the 2D Glauber–Ising model, is called for.

Acknowledgments

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